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Canonical general relativity: the primary constraint algebra

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Abstract. This is the second paper in a series discussing the canonical treatment of general relativity in vierbein formulation. We derive the primary constraints, and find that they satisfy an algebra the same as that of the generators of local Poincaré transformations, namely the Poincaré algebra.

1. Introduction

This paper is the second in a series in which we discuss the canonical treatment of general relativity in a vierbein basis. In the first paper (Charap and Nelson 1983) we derived the most general gravitational action that is free of second derivatives and, on variation, gives Einstein's equations. This action also does not require any additional boundary terms (see e.g. Gibbons and Hawking 1977, York 1972). In this paper we continue our investigation of the purely vierbein action (i.e. one that cannot be written in terms of metric variables) to determine the primary constraints and their algebra. We again consider pure gravity only, i.e. with no matter couplings. For non-derivative i.e. boson couplings, the results are unchanged. For fermion couplings there would be an extra contribution to the gravitational vierbein momenta from the spin coupling term in the action (see e.g. Nelson and Teitelboim 1978) but we expect the analysis to proceed along similar lines.

In § 2 we discuss the action that we use and the effect of the arbitrary foliation of space–time by a succession of three-dimensional hypersurfaces using proper time as our time variable. In § 3 we use this action to derive momenta and ten primary constraints, and their resulting algebra.

Our notation, conventions and signature $(+---)$ are the same as in Charap and Nelson (1983).

2. The vierbein action

Our starting point is the familiar Hilbert–Einstein expression for the action integral, namely

$$I(g)=\int_M R(g)\sqrt{-{}^4g}\,\mathrm{d}^4x$$

where $R(g)$ is the curvature scalar, ${}^4g$ is the determinant of the metric tensor $g_{\mu\nu}$, and M the space–time manifold. To this action should be added a surface integral over the boundary ∂M. An integration by parts removes the second derivatives of the vierbein fields $L_{a\mu}$ from

$$R = R_{ab\mu\nu} L^{a\nu} L^{b\mu}$$

with

$$R_{ab\mu\nu} = B_{\nu ab,\mu} - B_{\mu ab,\nu} + B_{\mu a}{}^{c} B_{\nu cb} - B_{\nu a}{}^{c} B_{\mu cb}$$

and

$$B_{\mu ab} = \tfrac{1}{2} L_b{}^{\alpha} L_a{}^{\nu} [L^c{}_{\nu}(L_{c\alpha,\mu} - L_{c\mu,\alpha}) - L^c{}_{\alpha}(L_{c\nu,\mu} - L_{c\mu,\nu}) + L^c{}_{\mu}(L_{c\alpha,\nu} - L_{c\nu,\alpha})]. \tag{2.1}$$

The result is to give

$$I = \int_M \sqrt{-{}^4 g}\, B_{\alpha ac} B_{\nu}{}^{c}{}_{b} (L^{a\alpha} L^{b\nu} - L^{a\nu} L^{b\alpha})\, \mathrm{d}^4 x \tag{2.2}$$

(where by $\sqrt{-{}^4 g}$ we now mean the determinant of $L_{a\mu}$, considered as a 4×4 matrix), or using (2.1)

$$I = \int_M \tfrac{1}{2} \sqrt{-{}^4 g}\, L_{a\mu,\nu} L_{b\lambda,\alpha} L_c{}^{\mu} L_d{}^{\nu} L_e{}^{\lambda} L_f{}^{\alpha} G^{abcdef}\, \mathrm{d}^4 x \tag{2.3}$$

with

$$\begin{aligned} G^{abcdef} = {} & \eta^{ab}(\eta^{ce}\eta^{df} - \eta^{de}\eta^{cf}) + 2\eta^{ac}(\eta^{de}\eta^{bf} - \eta^{df}\eta^{be}) - 2\eta^{ad}(\eta^{ce}\eta^{bf} - \eta^{cf}\eta^{be}) \\ & + \eta^{bc}(\eta^{df}\eta^{ae} - \eta^{de}\eta^{af}) - \eta^{bd}(\eta^{cf}\eta^{ae} - \eta^{ce}\eta^{af}). \end{aligned} \tag{2.4}$$

This numerical quantity G^{abcdef} possesses certain symmetries, namely

$$G^{abcdef} = G^{abcd[ef]} = G^{ab[cd]ef} = G^{baefcd}.$$

We will also make use of two important properties:

$$G^{abc(d|e|f)} = G^{(a|b|c)(d|e|f)} = G^{a(b|cd|e)f}$$

and

$$G^{abc[d|e|f]} = G^{[abc][def]} = \varepsilon^{abch} \varepsilon^{def}{}_{h}$$

where

$$\lambda_{(ab)} = \tfrac{1}{2}(\lambda_{ab} + \lambda_{ba}), \qquad \lambda_{[ab]} = \tfrac{1}{2}(\lambda_{ab} - \lambda_{ba}).$$

We have argued elsewhere (Charap and Nelson 1983) that when I is written in this form ((2.2) or (2.3)) there are no surface terms required. We assume that the manifold M is foliated by a succession of three-dimensional hypersurfaces Σ, the normals to which we will write as u, with $u^2 = +1$ for space-like Σ, and $u^2 = -1$ for time-like Σ. The surfaces Σ define at every point of M a proper time one-form $\mathrm{d}\tau$ dual to the normal u. We may write

$$\mathrm{d}\tau = u_{\mu}\, \mathrm{d}x^{\mu}$$

and it is with respect to this proper time variable τ that we will develop the canonical formulation, for space-like hyperspaces, i.e. $u^2 = +1$. For any field f over M we define

$$\dot{f} = \partial f / \partial \tau = f_{,\mu} u^{\mu}$$

and then

$$f_{,\nu} = u_{\nu} \dot{f} + (\delta_{\nu}^{\alpha} - u^{\alpha} u_{\nu}) f_{,\alpha}$$

shows how the gradient of f may be separated into components normal and tangential to Σ, this tangential component being

$$f_{,\hat{\nu}} = (\delta^{\alpha}_{\nu} - u^{\alpha}u_{\nu})f_{,\alpha}.$$

In particular we have

$$L_{a\mu,\nu} = u_{\nu}\dot{L}_{a\mu} + L_{a\mu,\hat{\nu}}$$

so that the action ((2.2) or (2.3)) has the form

$$I = \int_M \tfrac{1}{2}\sqrt{-{}^4g}\,(u_{\nu}\dot{L}_{a\mu} + L_{a\mu,\hat{\nu}})(u_{\alpha}\dot{L}_{b\lambda} + L_{b\lambda,\hat{\alpha}})L_c{}^{\mu}L_d{}^{\nu}L_e{}^{\lambda}L_f{}^{\alpha}G^{abcdef}\,\mathrm{d}^4x.$$

We see that this action is expressed as a functional of the vierbeins $L_{a\mu}$, their tangential derivatives $L_{a\mu,\hat{\nu}}$, the one-form field u_{α} and the vierbein velocities $\dot{L}_{a\mu}$. Our dynamical variables will therefore be the sixteen fields $L^{a}{}_{\mu}$, and their conjugate momenta.

3. Algebra of primary constraints

The four-dimensional volume element $\sqrt{-{}^4g}\,\mathrm{d}^4x$ may be written as $\mathrm{d}\Sigma\,\mathrm{d}\tau$ where $\mathrm{d}\Sigma$ is the volume element of the surface Σ. If γ is the metric in Σ induced by the metric g in M, then $\mathrm{d}\Sigma = \sqrt{\gamma}\,\mathrm{d}^3x$. Thus if

$$I = \int \mathscr{L}\,\mathrm{d}\tau$$

then the Lagrangian $\mathscr{L}$ is given by

$$\mathscr{L} = \int_{\Sigma} \tfrac{1}{2}\sqrt{\gamma}\,(u_{\nu}\dot{L}_{a\mu} + L_{a\mu,\hat{\nu}})(u_{\alpha}\dot{L}_{b\lambda} + L_{b\lambda,\hat{\alpha}})L_c{}^{\mu}L_d{}^{\nu}L_e{}^{\lambda}L_f{}^{\alpha}G^{abcdef}\,\mathrm{d}^3x. \quad (3.1)$$

From this Lagrangian (3.1) it follows immediately that conjugate to $L_{a\mu}$ is the momentum three-density

$$\pi^{a\mu} = \delta\mathscr{L}/\delta\dot{L}_{a\mu} = \sqrt{\gamma}\,(u_{\alpha}\dot{L}_{b\lambda} + L_{b\lambda,\hat{\alpha}})L_c{}^{\mu}u_dL_e{}^{\lambda}L_f{}^{\alpha}G^{abcdef}. \quad (3.2)$$

We observe that, from this definition, there are ten† primary constraints relating the $\pi^{a\mu}$'s and the $L_{a\mu}$'s. They consist of the six‡ generators of local vierbein rotations, defined by

$$J^{ab} = L^{a}{}_{\nu}\pi^{b\nu} - L^{b}{}_{\nu}\pi^{a\nu} - 2\sqrt{\gamma}\,\varepsilon^{acb}{}_{h}\varepsilon^{defh}L_{c\lambda,\hat{\alpha}}L_e{}^{\lambda}L_f{}^{\alpha}u_d \approx 0 \quad (3.3a)$$

† These ten primary constraints reduce the numbers of independent vierbein fields $L_{a\mu}$ from sixteen to six, corresponding exactly to the six g_{ij}'s of the ADM analysis (Arnowitt *et al* 1962). These six degrees of freedom will later be reduced to the two real degrees of freedom of the gravitational field, by the implementation of four secondary constraints corresponding to the four generators of general coordinate transformations.

‡ One checks that these six J^{ab} (3.3a) do indeed generate local rotations of the vierbein field by computing the bracket

$$[\tfrac{1}{2}\varepsilon_{ab}(x)J^{ab}(x), L^{c}{}_{\mu}(x')] = \varepsilon^{c}{}_{a}L^{a}{}_{\mu}\delta^3(x-x')$$

where $\varepsilon_{ab}(x)$ is an arbitrary local parameter. However, one also checks that

$$[\tfrac{1}{2}\varepsilon_{ab}(x)J^{ab}(x), \pi^{c\nu}(x')] \neq \varepsilon^{c}{}_{a}\pi^{a\nu}\delta^3(x-x').$$

and the four

$$T^a = \pi^{a\nu} u_\nu \approx 0. \tag{3.3b}$$

There are no further primary constraints. These primary constraints are to be added to the canonical Hamiltonian with arbitrary Lagrange multipliers to give the total Hamiltonian which is the generator of time evolution of the system.

Our next step is to use the canonical Poisson bracket relations between the dynamical field variables $L_{a\mu}$ and their conjugate momentum densities $\pi^{a\mu}$ (3.2) to determine the algebra of primary constraints. These Poisson brackets are of course between variables defined at points lying on the same surface Σ. If $\phi(x)$ is a field and $\pi(x')$ its conjugate field momentum, with x and x' both on Σ, then

$$[\phi(x), \pi(x')] = \delta^{(3)}(x - x')$$

where $\delta^{(3)}(x-x')$ is the distribution which satisfies

$$\int_\Sigma f(x')\delta^{(3)}(x-x')\,\mathrm{d}^3x' = f(x)$$

for any test function f. Thus the only non-vanishing canonical Poisson brackets are

$$[L_{a\mu}(x), \pi^{b\nu}(x')] = \delta_a{}^b \delta_\mu{}^\nu \delta^{(3)}(x-x')$$

and

$$[L_{a\mu,\hat\lambda}(x), \pi^{b\nu}(x')] = \delta^a{}_b \delta_\mu{}^\nu \delta^{(3)}{}_{,\hat\lambda}(x-x'). \tag{3.4}$$

Note that in (3.4) the derivative is along a direction indexed by $\hat\lambda$, i.e. tangential to Σ, so that the distribution on the right-hand side is well defined, and

$$\delta^{(3)}{}_{,\hat\lambda}(x-x') = (\partial/\partial x^{\hat\lambda})\delta^{(3)}(x-x') = -(\partial/\partial x'^{\hat\lambda})\delta^{(3)}(x-x').$$

It is of course a direct consequence of (3.3b) that

$$[T^a(x), T^b(x')] = 0. \tag{3.5}$$

The derivations of the other two brackets are less straightforward. The simpler one is

$$[T^c(x), J^{ab}(x')] = [T^c(x), L^{ab}(x')] + [T^c(x), S^{ab}(x')]$$

where

$$J^{ab} = L^{ab} + S^{ab}$$

with

$$L^{ab} = L^a{}_\nu \pi^{b\nu} - L^b{}_\nu \pi^{a\nu}, \qquad S^{ab} = -2\sqrt{\gamma}\,\varepsilon^{acb}{}_h \varepsilon^{defh} L_{c\lambda,\hat\alpha} L_e{}^\lambda L_f{}^\alpha u_d$$

or, more conveniently,

$$S^{ab} = 4(\sqrt{\gamma} L^{[a|\alpha|} u^{b]})_{,\hat\alpha}.$$

We find that

$$[T^c(x), L^{ab}(x')] = (T^a \delta^{bc} - T^b \delta^{ac})\delta^{(3)}(x-x')$$

and

$$[T^c(x), S^{ab}(x')] = 0 \tag{3.6}$$

so that

$$[T^c(x), J^{ab}(x')] = (T^a\delta^{bc} - T^b\delta^{ac})\delta^{(3)}(x - x'). \tag{3.7}$$

In deriving (3.6) we have used the relations

$$\delta(u_\nu\sqrt{\gamma})/\sqrt{\gamma} = u_\nu\delta(\sqrt{-{}^4g})/\sqrt{-{}^4g} \tag{3.8}$$

with

$$\delta u_\nu/\delta L_{a\mu} = 0$$

and the identity

$$f(x')(\partial/\partial x'^{\hat{\alpha}})\delta^{(3)}(x - x') = -f_{,\hat{\alpha}}\delta^{(3)}(x - x'). \tag{3.9}$$

This identity is valid when the distributions involved act on test functions and are integrated, i.e.

$$\int_\Sigma f(x')(\partial/\partial x'^{\hat{\alpha}})\delta^{(3)}(x - x')\,\mathrm{d}^3x' = -\partial f(x)/\partial x^{\hat{\alpha}}.$$

The derivation of the bracket $[J^{ab}(x), J^{a'b'}(x')]$ proceeds along similar lines, using (3.8) and (3.9). The result is

$$[J^{ab}(x), J^{a'b'}(x')] = (\eta^{ab'}J^{a'b} - \eta^{ba'}J^{ab'} + \eta^{bb'}J^{aa'} - \eta^{aa'}J^{b'b})\delta^3(x - x'). \tag{3.10}$$

4. Conclusions

The equations (3.5), (3.7) and 3.10) comprise the primary constraint algebra of general relativity in the vierbein basis. The primary constraints (3.3*a*), (3.3*b*) satisfy the same algebra as the generators of local Poincaré transformations, i.e. the Poincaré algebra. However, these primary constraints and their algebra should be viewed with caution, because they cannot necessarily be identified with the Poincaré generators, as can be seen by computing the brackets

$$[\tfrac{1}{2}\varepsilon_{ab}J^{ab}, L^c{}_\mu], [\tfrac{1}{2}\varepsilon_{ab}J^{ab}, \pi^{c\nu}], \qquad [\xi_aP^a, L^c{}_\mu], [\xi_aP^a, \pi^{c\nu}],$$

for arbitrary parameters ε_{ab} and ξ_a.

These primary constraints (3.3*a*), (3.3*b*) are to be added to the canonical Hamiltonian with arbitrary multipliers to give the total Hamiltonian H. The canonical Hamiltonian is defined as

$$H_{\mathrm{can}} = \int_\Sigma (\pi^{a\mu}\dot{L}_{a\mu} - \mathscr{L})\,\mathrm{d}^3x$$

and can be computed as a functional of $L_{a\mu}$ and $\pi^{a\mu}$ only, by elimination of velocities in favour of momenta, i.e. although there are only twelve independent momenta $\pi^{a\mu}$ because of (3.3*b*), and sixteen velocities $\dot{L}_{a\mu}$, their relationship (3.2) can indeed be inverted.

The total Hamiltonian H then obtained is used to compute secondary constraints, i.e. time derivatives of primary constraints by

$$\dot{F} = [F, H]$$

for functionals F of the canonical variables $L_{a\mu}$, $\pi^{a\mu}$.

Indeed, the four generators of general coordinate transformations appear as secondary constraints. The derivation of the total Hamiltonian and the secondary constraints will be given in the next paper of this series.

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